Table 2 y^+ v_s u^+ for inner layer

	Term I		Term II		u^+	
y^+	Eq. (3)	Eq. (6)	Eq. (3)	Eq. (6)	Eq. (3)	Eq. (6)
4	7.495	7.532	−0.049	0.219	3.926	3.942
10	8.575	8.528	3.347	3.655	8.402	8.374
20	9.245	9.209	5.901	6.186	11.626	11.589
30	9.702	9.686	6.830	7.118	13.012	12.994
40	10.093	10.090	7.281	7.574	13.852	13.854
50	10.435	10.440	7.544	7.842	14.459	14.472
100	11.686	11.710	8.042	8.358	16.215	16.257

Conclusion

Galbraith and Head's expression represents the velocity profiles very well. All of the four models overestimate the velocities in the outer part of the outer zone for relaxing flows. Musker and Liakopoulos' expressions practically predict the same values in the inner zone. Whitfield's expression for the inner zone is of very simple form.

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Prediction of Radially Spreading Turbulent Jets

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Introduction

ALTHOUGH the two-equation k-ϵ turbulence model has been widely used for the prediction of various jet flows, a systematic assessment of its performance for radially spreading jets appears not to have been made. The radial free jet has been computed by Rubel[1] and the radial wall jet by Sharma and Patankar.[2] The half-width spreading rates computed by these authors are compared with those of experiment[3–6] in Table 1. Although the predictions appear satisfactory, it is important to note that while Rubel employed the standard k-ϵ model,[7] Sharma and Patankar used a different set of values for the model coefficients.

In the present work, the standard k-ϵ model is used to predict both types of radial wall jets. Some attention is also given to the prediction of plane and round jets in view of their importance as benchmarks for turbulence-model development. Calculations are made with the standard set of model coefficients, and the performance of the model is assessed by comparing the predicted spreading rates with experimental data. It will be shown in the results section that the standard model tends to underestimate the measured spreading rates of radial jets, especially for the wall-jet configuration. The discrepancies between prediction and experiment are attributed to deficiences in the ϵ-transport equation. In this study, a modification is made to this equation which results in improved predictions of radial jets.

Turbulence-Model Equations

The standard k-ϵ model determines the Reynolds stresses $-\overline{u_iu_j}$ through the isotropic eddy-viscosity hypothesis

$$-\overline{u_iu_j} = \nu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}\,k\,\delta_{ij} \tag{1}$$

where U_i is the mean velocity component in the x_i direction, δ_{ij} is the Kronecker delta, k is the turbulent kinetic energy, and ν_t is the kinematic eddy viscosity, which is given by

$$\nu_t = C_\mu\, k^{1/2}\, L \tag{2}$$

where L is the macro length scale of the turbulent motion. The k-ϵ model determines k and L from semiempirical transport equations for k and the second turbulence property ϵ, the rate of dissipation of k. The length scale is recovered from the local values of k and ϵ by way of

$$\epsilon = C_D\, k^{3/2}/L \tag{3}$$

The turbulent kinetic energy is calculated from the following transport equation:

$$\rho\frac{\mathrm{D}k}{\mathrm{D}t} = \frac{\partial}{\partial x_i}\left(\frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial x_i}\right) + \rho\,(P_k - \epsilon) \tag{4}$$

where μ_t is the dynamic eddy viscosity and P_k is the rate of production of k,

$$P_k = -\overline{u_iu_j}\,\frac{\partial U_i}{\partial x_j} \tag{5}$$

The dissipation rate is calculated from the following transport equation:

$$\rho\,\frac{\mathrm{D}\epsilon}{\mathrm{D}t} = \frac{\partial}{\partial x_i}\left(\frac{\mu_t}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial x_i}\right) + \rho\, C_1\frac{\epsilon}{k}\,P_k - \rho\, C_2\,\frac{k^{1/2}}{L}\,\epsilon \tag{6}$$

The six empirical coefficients are assigned the values recom-

Table 1 Previous work: calculated and measured jet spreading rates

	Spreading rate, $\mathrm{d}\delta_U/\mathrm{d}x$	
Jet	Calculated (k-ϵ)	Data
Radial wall	0.080	0.085–0.095
Radial free	0.095	0.098–0.110

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mended by Rodi[7] as follows: $C_\mu = 0.09$, $C_1 = 1.44$, $C_2 = 1.92$, $C_D = 1.0$, $\sigma_k = 1.0$, and $\sigma_\epsilon = 1.314$. The mean-flow and turbulence-model equations are parabolized in the main-flow direction and solved numerically with a forward-marching, finite-volume procedure.[8] The marching-integration process is carried out until self-similarity is attained. The wall function approach outlined by Rodi[7] is used to bridge the viscous sublayer adjacent to walls. The wall shear stress is obtained from the well-known law of the wall in which von Kàrmàn's constant is taken to be 0.41 and the roughness parameter as 8.6 for smooth walls.

Turbulence-Model Modifications

It is known that when the standard k-ϵ model is applied[9] to the plane wall jet, the spreading rate is calculated to be about 25% greater than experiment. Therefore, it is somewhat surprising to find that the same model should underestimate the spreading rate of the corresponding radial case by almost the same amount. The plane wall jet can be predicted correctly by introducing the algebraic stress model (ASM) of Ljuboja and Rodi,[9] which accounts for the wall-damping process in wall jets. Clearly this mechanism is still present in the radial wall jet, but it is easy to see that the introduction of such a model would result in worse agreement with experiment, as its influence is to reduce the lateral transfer of momentum due to turbulence. Calculations are presented in the results section demonstrating the failure of the ASM for the radial wall jet. From these findings it is apparent that the disagreement with data is due to some other component of the k-ϵ model.

If one assumes that plane and radial jets spread at the same rate, then similarity theory can be used to show that the shear-stress level in the radial case should be twice as high as that in the plane case. Experiments confirm this expectation, and so it follows from Prandtl's hypothesis that L should be larger by a factor of $\sqrt{2}$ for the radial case. However, numerical calculations made by the present author indicate that the computed increase in length scale between the two flows is underestimated. This suggests that the level of ϵ is too low for the radial case, giving too low energy and shear-stress levels. This situation is remedied by introducing a modification that reduces the generation rate of ϵ so as to promote scale augmentation. The modification is based on Hanjalic and Launder's[10] (cited hereafter as HL) preferential treatment of normal stresses in the scale equation.

In general, a modification is needed to the standard k-ϵ model whereby ϵ is reduced for the case of a radial jet but not for a plane or round jet. The distinction between the various flows can be made by noting that lateral divergence of the flow is present in radial jets but not in plane or round jets. Bradshaw[11,12] has commented on the surprisingly large effect lateral divergence can have on the turbulence structure. For example, experiments indicate that lateral divergence leads to an accentuation of the large-scale eddies and an increase in the entrainment rate. The suggestion made here is that the k-ϵ model should be modified so that the extra rate of strain associated with lateral divergence has a significant influence on the process of scale augmentation. Therefore, it was decided to include the irrotational strain terms, which had hitherto been omitted, into the production term P_k. These terms are generally neglected in parabolic-flow simulations. When these secondary generation terms are included, the production rate of k is given by

$$P_k = -\overline{uv}\,\frac{\partial U}{\partial y} - (\overline{u^2} - \overline{v^2})\,\frac{\partial U}{\partial x} + (\overline{v^2} - \overline{w^2})\,\frac{U}{x} \tag{7}$$

where the first contribution is the usual rotational term and the second and third terms are, respectively, irrotational contributions due to longitudinal deceleration and lateral divergence. The term involving $\partial U/\partial x$ is included after HL because of its importance in round jets. It is also likely to be as influential in the radial case because for this flow the velocity decays at the same rate as in the round jet. The deceleration term can be either positive or negative, as explained by HL. In the outer regions of the jet, the term is likely to be positive, leading to a reduction in P_k, whereas in the vicinity of the velocity maximum, it is likely to be negative, leading to an amplification of P_k. However, the extra production of k due to lateral divergence is always negative in radial jets and zero in plane and round jets. Thus, while the deceleration term may create higher or lower levels of ϵ, the lateral-divergence term can be expected always to produce a decrease in ϵ.

As the k-ϵ model does not provide values of the normal stresses explicitly, they are specified empirically in terms of k as follows:

$$(\overline{u^2} - \overline{v^2}) = \alpha k; \qquad (\overline{v^2} - \overline{w^2}) = \beta k \tag{8}$$

where α and β are empirical constants. Experimental data[3] on various free-jet flows suggest representative values of $\alpha = 0.35$ and $\beta = -0.06$, whereas various wall-jet experiments[4,13] suggest values of $\alpha = 0.42$ and $\beta = -0.11$. These changes in α and β from the free case to the wall flow reflect the damping effect of a wall on the lateral velocity fluctuations. In future studies, α and β could be calculated from an ASM so as to remove this empiricism.

As direct application of Eq. (7) did not significantly alter the predicted spreading rates, the following practice was employed after HL. Equation (7) was retained in the k equation, but in the ϵ equation, the irrotational contributions were multiplied with coefficients that differ from the one multiplying the rotational part. The source term in the modified ϵ-equation then reads

$$\left(-C_1\overline{uv}\,\frac{\partial U}{\partial y} - C_4(\overline{u^2} - \overline{v^2})\,\frac{\partial U}{\partial x} + C_5(\overline{v^2} - \overline{w^2})\,\frac{U}{x}\right)\rho\,\frac{\epsilon}{k} - \rho\, C_2\,\frac{k^{1/2}}{L}\,\epsilon \tag{9}$$

wherein the empirical coefficients C_4 and C_5 are given values of 4.2 and 11.0, respectively. In order to improve the shape of the velocity and turbulence energy profiles toward the edges of the jet, the diffusion coefficient σ_ϵ is set equal to 1.1.

Results and Discussion

The results are presented in terms of the half-width spreading rates. Computed results are compared with experiment in Table 2 for five types of jet flow. The table includes predictions for three different model types: the standard model, the ASM (for wall jets only), and the modified model, which uses normal-stress amplification (NSA) in the ϵ equation (for all cases except the plane wall jet). More detailed comparisons with experimental data have been reported by the author elsewhere.[14]

Table 2 Comparison of calculated and measured jet spreading rates

		Spreading rate, $d\delta_U/dx$	
Jet	Model type	Calculated (r, k-ϵ)	Data
Radial wall	Standard	0.068	0.085–0.095
	Modified (ASM)	0.054	
	Modified (NSA)	0.089	
Radial free	Standard	0.090	0.098–0.110
	Modified (NSA)	0.100	
Round free	Standard	0.113	0.086
	Modified (NSA)	0.098	
Plane free	Standard	0.104	0.100–0.110
	Modified (NSA)	0.115	
Plane wall	Standard	0.093	0.071–0.075
	Modified (ASM)	0.075	

For radial wall jets, Table 2 shows that the standard k-ϵ model produces a spreading rate about 20% below the recommended band of experimental values.[13] For the radial free jet, it can be seen that the model performs somewhat better, as it yields a spreading rate about 8% below the lowest measured value. It is not entirely clear why the present set of results differ from those shown in Table 1. For the radial wall jet, the differences may arise from the fact that Sharma and Patankar[2] used a different set of values for the model coefficients, and also that their spreading rate was deduced by the present author from a very small published graph, a process that inevitably introduces some error. For the radial free jet, Rubel[1] calculated a growth rate 5% in excess of the present calculations using the same k-ϵ model but with a different numerical technique.[15] On the basis of his results it may be argued that the k-ϵ model adequately represents the growth rate of 0.098 measured by Tanaka and Tanaka.[5] However, it is evident that the model predictions are still someway below the respective values of 0.106 and 0.11 measured by Witze and Dwyer[6] and Heskestad.[16]

The plane free jet represents a reliable bench mark by which the accuracy of the present calculation scheme can be judged. For this case, Table 2 shows that the standard k-ϵ model produces a growth rate of 0.104, which is 4% below the value of 0.108 set by Paullay et al.[15] as the standard for parabolic-type calcuations. Other workers[9,10] using parabolic-type marching schemes have computed growth rates in the range of 0.109–0.114.

Table 2 shows that when the ASM[9] is applied to the radial wall jet, the predicted spreading rate is reduced further, so that the agreement with experiment is now worse than that found with the original k-ϵ model. This result is a consequence of the fact that with the ASM the level of shear stress is reduced by the wall-damping model, leading to a decrease in the rate of spread. In contrast, Table 2 shows that the ASM gives very good agreeement with the data for the plane wall jet. If NSA is introduced for this case, the agreement deteriorates somewhat because the spreading rate is increased by about 10%.

The attention is now focused on the results obtained with the modified model that employs NSA. As was previously demonstrated by HL, Table 2 shows that the modified model increases the growth rate of the plane free jet and gives substantial improvement in the calculated rate of spread for the round jet. Turning now to the radial jets, Table 2 shows that the k-ϵ model with NSA predicts generally satisfactory agreement with the measured spreading rates of both types of radial jet. As expected, the introduction of NSA results in an increase in L, leading to greater k levels and greater eddy viscosities and so to an increase in the jet spreading.

Conclusions

The k-ϵ model was modified so that the production rate of k associated with the lateral divergence of the radial flow has a significant influence on the process of scale augmentation. It was demonstrated that the modified model yielded much-improved predictions of radial jets. Further work should consider determining the normal-stress components from an ASM.

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Quadrature Formula for a Double-Pole Singular Integral

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Introduction

In linearized potential thin-wing and airfoil theory, one encounters integrals of the following two forms:

$$I_1(x) = \int_{-1}^{1} (\xi - x)^{-1} f(\xi)\, d\xi, \qquad -1 < x < +1 \tag{1}$$

$$I_2(x) = \int_{-1}^{1} (\xi - x)^{-2} f(\xi)\, d\xi, \qquad -1 < x < +1 \tag{2}$$

In the conventional (Riemann) sense, these integrals are meaningless because of the pole singularity at $x = \xi$. For the origin and context of these important integrals, one may refer to Mangler,[1] who also provides rules for their appropriate interpretation.

The value of these integrals, when they exist, is called the principal value of the concerned integral, when they are interpreted, respectively, in the following limiting sense:

$$\int_{-1}^{1} (\xi - x)^{-1} f(\xi)\, d\xi = \operatorname*{Lim}_{\epsilon \to 0} \left[\int_{-1}^{x-\epsilon} (\xi - x)^{-1} f(\xi)\, d\xi \right.$$

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